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The mixing length hypothesis in the turbulence theory

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Abstract—Main landmarks in the evolution of the mixing length hypothesis, its impact on the development of calculation methods for the turbulent boundary layers, and the general influence on the development of the turbulence theory are considered.

THE YEAR 1983 marked the centenary of the publication of Reynolds' paper [1] on the results of the experiments in which, for the first time, the existence of two main flow regimes and the effect of their transition from one to the other had been clearly demonstrated with the aid of a tinted jet. The year 1985 marks 90 years since the publication of another paper of his in which, also for the first time, the equations of the averaged turbulent flow were formulated and the problem set to determine the critical value of what is now called the Reynolds number [2]. The incorporation into these equations of the terms that characterize the molar transfer effects, i.e.

$$\tau_{ji} = -\rho\langle u_j' u_i' \rangle \tag{1}$$

has set forth the problems of the closure of the equations and of the formulation of corresponding boundary conditions. The higher-order moment equations set up for the first time by Keller and Friedman [3] distinctly demonstrated the fundamental nature of these problems in a most general formulation.

The mixing length hypothesis, owing its origin to Prandtl [4, 5] and Taylor [6], was the first, and surprisingly successful, attempt to close the Reynolds equations for a two-dimensional boundary layer. In the author's opinion, the influence of this hypothesis on the development of the theory, experimental research and engineering calculation methods of the turbulent boundary layers cannot be overemphasized. Whilst its heuristic aspect and terminology were time and again subjected to scepticism and argument, in actual fact its impact on the development of the turbulence theory lies far outside the framework of specific applications. It is also a remarkable fact that the mixing length concept is closely connected with the development of physical and mathematical models on the basis of the physical similarity analysis and has actually been advanced at the same time as the foundations of the latter have been laid [7–10].

The limiting hydrodynamic situation, in which there is a simple interrelation between the averaged moments of the fluctuating flow velocity vector component and the averaged velocity, has been most lucidly formulated by Prandtl.

Suppose a one-dimensional averaged flow on an infinite smooth plate is considered, whose averaged velocity $\langle u \rangle$ changes only along the normal to the solid surface. In a model such as this, which is asymptotic with respect to the actual flows, the situation is characterized by the following conditions

$$\{y = 0,\ \langle u \rangle\ 0,\ u_i' = 0;\quad y \to \infty,\ \langle u \rangle \to U$$
$$\operatorname{grad} p = 0;\quad \tau = \tau_w;\quad \langle u \rangle = f(y)\}. \tag{2}$$

Moreover, there are two physical properties of the medium which are essential to the physical problem considered: the density ρ and the dynamic viscosity of the liquid μ. The condition $\mu > 0$ is responsible for the effect of liquid clinging to the solid wall and, consequently, for the appearance of a certain wall layer in which the effect of molecular friction makes itself felt appreciably, while the turbulent friction is negligibly small. Prandtl called this region $(0 < y < y_1)$ the laminar sublayer. Outside this sublayer $(y > y_1)$, the averaged flow is virtually entirely governed by the turbulent transfer in the sense of equation (1).

Just like Reynolds, Prandtl started from the analogy with the kinetic theory concepts having assumed the magnitude of turbulent disturbance to be proportional to the averaged velocity gradient and to some linear scale being an analog of the molecular free path length

$$u_i' = l_k \frac{\partial \langle u_i \rangle}{\partial x_k}. \tag{3}$$

The quantity l_k has been termed the mixing length, i.e. a certain distance at which the appearing turbulent disturbances (called 'vortices' by Taylor and 'moles' by Prandtl) lose their individuality.

In the limiting Prandtl model formulated above, there is only one characteristic linear dimension, so that, apart from the sign,

$$\langle u'v' \rangle = \left(\aleph y \frac{\mathrm{d}\langle u \rangle}{\mathrm{d}y} \right)^2 \tag{4}$$

where $\aleph$ is a certain constant.

Note should be taken of the following important fact: the derivation of formula (3) in this model requires no analogies at all with the kinetic theory of heat and can be carried out in a purely formal way (or, to be more precise, in a most abstract manner) proceeding only from the dimensional reasoning and from a clear

NOMENCLATURE

c	heat capacity
C_{fo}	friction factor of a smooth impermeable surface
L	characteristic dimension of the streamlined body
p	pressure
q_y	turbulent heat flux vector component normal to the wall
T'	temperature fluctuation
u'_i	ith component of the fluctuating flow velocity vector
$\langle u'_i u'_j \rangle$	one-point correlation of velocity fluctuations
u^+	dimensionless velocity, $\langle u \rangle / u^*$
$\langle \bar{u} \rangle$	relative averaged velocity, $\langle u \rangle / U$
v'	normal component of velocity fluctuations
y	coordinate normal to the surface
y^+	dimensionless coordinate, $(u^* y)/\nu$
$\bar{y}$	relative transverse coordinate, y/δ

Greek symbols

δ	characteristic thickness of the boundary layer
ν	kinematic viscosity, μ/ρ
ρ	density of the fluid
$\bar{\rho}$	relative gas density, ρ/ρ_0
ρ_0	gas density outside the boundary layer
τ_{ij}	Reynolds stresses
τ	shearing stress in the plane (x, z)
τ_w	stress on a solid surface
$\bar{\tau}$	relative shearing stress, τ/τ_w
Ψ	relative friction factor, C_f/C_{fo}

physical concept that in the flow considered the source of turbulent disturbances is the flow friction against the wall which causes the inhomogeneous averaged flow velocity profile.

In this way, the first two-layer gradient model of a turbulent boundary layer appeared which made it possible to find the famous law which governs the logarithmic velocity distribution

$$y^+ > y_1^+, \quad u^+ = C + \frac{1}{\aleph} \ln y^+. \qquad (5)$$

Here $C = y_1^+ - \aleph^{-1} \ln y_1^+$. The quantity $u^* = \sqrt{\tau_w/\rho}$ is a certain intrinsic scale for the velocity in the given model commonly referred to as the friction velocity; y_1^+ is the conditional edge of the laminar sublayer given by the intersection of the logarithmic velocity profile, equation (5), in the region $y^+ > y_1^+$ and the linear velocity profile $u^+ = y^+$ in the region $y < y_1^+$. The experiments by Nikuradze [11] brilliantly confirmed the main results of the mixing length model. However, the problem of describing the turbulent transport of heat (and generally of any admixtures to the liquid) even in an extremely simplified physical model such as this required that what is transported by the turbulent velocity fluctuations be more clearly determined as

$$q_y = -c_p \langle T'v' \rangle. \qquad (6)$$

On Prandtl's model, the transports of momentum and heat by formulae (1) and (6) turned out to be completely identical, i.e. the 'turbulent Prandtl number' $Pr_T = 1$. On Taylor's vorticity transport model, $Pr_T < 1$ and, as was shown by Taylor, for a turbulent wake this model gave $Pr_T = 0.5$. The experiment supported this conclusion rather well, and thus the first fundamental foundation of the eddy turbulence models was set down.

Prandtl's works already clearly pointed to a complex hierarchal structure of the turbulent flow and to the fact that the mixing length should be considered as the characteristic turbulent transfer scale being dependent on a number of factors. In particular, this is evidenced by his remarks about turbulent transports on the channel axis, where $\partial\langle u \rangle/\partial y = 0$ and a purely gradient description such as equation (3) are inadequate, and about the existance of the flow regions in which the mixing length is of the order of the characteristic linear dimension as, e.g. in free submerged jets and in wall boundary layers at $y \gtrsim 0.2\delta$, where δ is the channel half-width or the nominal boundary layer thickness in an infinite flow.

In a certain sense, the formula

$$l = \aleph y \qquad (7)$$

also reflects the 'long-range' turbulence effects, i.e. the dependence of the local turbulence parameters not only on the local averaged velocity profile characteristics (in a simple model on $\partial\langle u \rangle/\partial y$), but also on the mixing length determined integrally. It is not by chance that the possibility arises for an integral description of a number of turbulent flows in terms similar to those used in the radiation theory [12]. However, the quest for a local description of turbulence was stimulated not only by certain traditions in the hydrodynamics, but also by an intuitive desire of some investigators to make these descriptions more uniform. This trend is associated with work by Karman [13] and its extension by Loitsyansky [14], Novozhilov [15] and a number of other scientists. The respective hypothesis was formulated by von Karman thus: '... the similarity of fluctuations irrespective of the point in the flow in whose vicinity these fluctuations are being considered'. The basic and practically usable result of this hypothesis is the following expression for the mixing length in a two-dimensional boundary layer

$$l = \aleph \frac{\partial \langle u \rangle}{\partial y} \bigg/ \frac{\partial^2 \langle u \rangle}{\partial y^2}. \qquad (8)$$

For the conditions $\rho = \text{const.}$ and $\tau = \text{const.}$, this formula yields the same result as Prandtl's formula (7). The numerical values of the constants $\aleph$ are also the same in these formulae. However, in more complex situations (nonisothermicity, compressibility, permeability of the streamlined surface, pressure gradient, etc.) the Prandtl and von Karman hypotheses yield ambiguous results. For example, one may compare the final results of refs. [16] and [17]. In the author's opinion, the whole background of actual information points to the limited usefulness of the local models as compared with the integral ones.

The extension of the mixing length model to spacial flows turns out to be quite effective in a number of cases. Among the examples of such extensions are the works of Buleyev [18].

It was von Karman who, within the framework of the mixing length hypothesis, made an important remark on the great significance of the transition region between the laminar sublayer and the region of fully developed turbulence [19]. For an incompressible liquid boundary layer at small pressure gradients this region is roughly confined within $5 < y^+ < 50$. Here, both the molecular and turbulent friction effects are commensurable, varying from the predominant influence of the former when $y^+ \to 5$–10 to the full predominance of the latter at $y^+ \approx 30$–50. Exhibiting a slight effect in the hydrodynamic friction calculations, this region substantially influences the heat transfer rate at the Prandtl numbers of the order of 5–25. At still greater Pr numbers, the turbulent transfer of heat in the laminar sublayer becomes operative, so that the following condition is valid [20, 21]:

$$y^+ < y_1^+,\ Pr \to \infty;\quad Pr_T \to \text{const.}\, Pr^{-1}. \tag{9}$$

The reason for this is that in a viscous sublayer the turbulent viscosity $\mu_T \sim y^3$, while the turbulent thermal conductivity $\lambda_T \sim y^n$, so that the exponent n varies from 3 to 4 with an increasing Pr number.

Von Karman's logarithmic approximation of the velocity profile in the buffer sublayer ($5 < y^+ < 50$) has turned out to be effective for a number of calculations, including the non-Newtonian media, and besides in a much wider range of the y^+ values (for example, a channel can be completely filled with such a 'buffer' flow alone).

It can be said with a certain confidence that ever since von Karman's works great interest has developed in the study of turbulent effects in the immediate vicinity of the solid wall, and the change-over has started from the original term 'laminar sublayer' to a more substantial concept 'viscous sublayer' with its complex and strong turbulence [22–25].

The examples of efficient programmes for boundary layer calculations on the basis of the mixing length hypothesis can be found in ref. [26].

It is thought that, to a certain degree, the classical ideas of the mixing length hypothesis have logically culminated in the model of a liquid with vanishing viscosity [27]. In a medium with an arbitrarily small, but never zero, viscosity, the following conditions hold for any flow velocities and dimensions of a streamlined body

$$\mu \gtrapprox 0: Re \to \infty,\quad \delta/L \to 0,\ y_1/\delta \to 0. \tag{10}$$

Thus, in this limiting (asymptotic) model, two general ideas of the mixing length and boundary layer hypotheses—the local one-dimensionality and large Reynolds numbers—are employed. Of principal importance also, is, the effect of viscous sublayer degeneration. This model leads to the integral relation of the form [20, 27]:

$$Re \to \infty,\quad \int_0^1 \left(\frac{\bar{\rho}}{\Psi \bar{\tau}_{\bar{y} \ll 1}}\right)^{1/2} \mathrm{d}\langle \bar{u} \rangle \to 1. \tag{11}$$

An essential feature of this integral relation is that in a number of canonical cases it can be solved in quadratures which makes it possible to select relatively simple theoretical formulae convenient for the programmes of calculations of many practically important problems [28].

As an example the following relations can be given:

(a) In the case of an incompressible isothermal flow past a permeable plate

$$Re \to \infty,\quad \Psi \to \left(1 - \frac{b}{4}\right)^2 \tag{12}$$

where $b = \bar{j}_w/C_{f0}$ is the permeability factor; $\bar{j}_w = j_w/\rho_0 U_0$ is the relative flow rate through a permeable surface.

(b) For a gas flow past an adiabatic plate

$$Re \to \infty,\ \psi = \psi^*;\quad \Psi \to \frac{\left(\arcsin\sqrt{\dfrac{\psi^*-1}{\psi^*}}\right)^2}{\psi^*-1} \tag{13}$$

where $\psi = T_w/T_0$ is the temperature factor; $\psi^* = 1 + \frac{1}{2}(\gamma - 1)M^2$ is the kinetic temperature factor; γ the Poisson specific heat ratio, M the Mach number.

(c) For a subsonic flow past a plate

$$Re \to \infty,\quad M \ll 1,\quad \Psi \to \left(\frac{2}{1+\sqrt{\psi}}\right)^2. \tag{14}$$

When the three factors b, ψ, and ψ^* interact, the relatively complex quadratures can be roughly approximated by the following simple formula

$$Re \to \infty,\quad \Psi \approx \Psi_b \Psi_{\psi^*} \Psi_{\psi}. \tag{15}$$

The subscripts used here correspond to formulae (12), (13) and (14).

A purely gradient description of turbulent transfer, the first example of which is given by formula (3), is inadequate for representing complex flows and the domain of extreme velocity, temperature, and concentration fields. This problem was first set by Prandtl within the framework of the mixing length model. For a qualitative description of turbulence transport in the region of the channel axis he suggested that the rate of turbulent transfer be evaluated on the basis of the velocity gradient averaged in the vicinity of its extremum

$$\langle u'v' \rangle \sim \left\langle \left(\frac{\partial \langle u \rangle}{\partial y} \right)^2 \right\rangle_{\Delta y}. \tag{16}$$

Here, Δy means that the averaging has been carried out in the neighbourhood of $\pm \Delta y$ from the coordinate y_0, which corresponds to the extremal value of $\langle u \rangle$.

CONCLUSION

The mixing length hypothesis turned out to be the first semiempirical theory of closure of the averaged turbulent flow equations. Its development gave rise to such fundamental concepts as:

—linear scale of turbulent transfer;
—separation of a turbulent flow into a number of interacting but distinct regions;
—vortical mechanism of turbulence transport;
—gradient and non-gradient mechanisms of turbulence transport;
—non-similarity between the processes of momentum and neutral impurity (specifically heat) transport;
—complex picture of the actual situations in the immediate neighbourhood of the solid wall (viscous sublayer) in the case of a quasilaminar averaged flow.

The theory has formed the basis for the development of a number of asymptotic, physically meaningful models; has given impetus for performing considerable amount of fundamentally important experimental research to investigate the averaged turbulent flows and also their thin structure; and has made it possible to develop efficient, well-tried (experimentally and practically) methods of calculation for numerous turbulent boundary layers and jet flows.

Thus, the mixing length hypothesis is the next, after the Reynolds equations, fundamental contribution to the theory to the theory of the developed turbulent flow.

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L'HYPOTHESE DE LA LONGUEUR DE MELANGE DANS LA THEORIE DE LA TURBULENCE

Résumé—On considère les points principaux de l'évolution de l'hypothèse de la longueur de mélange, son impact sur le développement des méthodes de calcul pour les couches limites turbulentes et l'influence générale sur le développement de la théorie de la turbulence.

DIE MISCHUNGSWEG-HYPOTHESE IN DER THEORIE DER TURBULENZ

Zusammenfassung—Betrachtet werden die wesentlichen Marksteine bei der Entwicklung der Mischungsweg-Hypothese, ihr Einfluß auf die Berechnungsverfahren für turbulente Grenzschichten und der allgemeine Einfluß auf die Entwicklung der Theorie der Turbulenz.

ГИПОТЕЗА ПУТИ СМЕШЕНИЯ В ТЕОРИИ ТУРБУЛЕНТНОСТИ

Аннотация—Рассматриваются существенные этапы развития гипотезы пути смешения, ее влияние на создание методов расчета турбулентных пограничных слоев, общее воздействие на развитие теории турбулентности.